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Singular integral operators for an unlimited contour

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Abstract

Let Γ be a closed or unclosed unlimited contour, a shift $\alpha(t)$ maps homeomorphically the contour Γ onto itself with preserving or reversing the direction on Γ and also satisfies the conditions for some natural number $n \geqslant 2$, $\alpha_n(t) \cong t$, and $\alpha_j(t) \ncong t$ for $1 \leqslant j < n$. In this work we study subalgebra Σ of algebra $L(L_p(\Gamma, \rho))$, which contains all operators of the form

$$(M\varphi)(t) = \sum_{k=0}^{n-1} \left\{ a_k(t)\varphi\big(\alpha_k(t)\big) + \frac{b_k(t)}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - \alpha_k(t)}\, d\tau \right\}$$

with piecewise continuous coefficients. The existence of such isomorphism between Σ and some algebra $\mathfrak{A}$ of singular operators with Cauchy kernel that an arbitrary operator from Σ and its image are Noetherian or not Noetherian simultaneously is proved. It allows to introduce the concept of a symbol for all operators from Σ and, using the known results for algebra $\mathfrak{A}$, in terms of a symbol to receive conditions of Noetherian property.

MSC: 45E05

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1. Introduction

Let Γ be a closed or unclosed unlimited contour, a shift $\alpha(t)$ maps homeomorphically the contour Γ onto itself with preserving or reversing the direction on Γ and also satisfies the conditions for some natural number $n \geqslant 2$ (if α reverses the orientation on Γ, n is always equal to two), $\alpha_n(t) \cong t$, and for $1 \leqslant j < n$, $\alpha_j(t) \ncong t$ $(t \in \Gamma; \alpha_j(t) = \alpha[\alpha_{j-1}(t)],\ j = 1, \dots, n-1, \alpha_0(t) \cong t)$;

$$\frac{\alpha_i'(t)(t - z_0)^2}{(\alpha_i(t) - z_0)^2} \in H(\Gamma) \quad (z_0 \in \mathbb{C} \setminus \Gamma). \tag{1.1}$$

The class of such functions will be designated by $V(\Gamma)$. Obviously, $V(\Gamma)$ does not depend on the choice of a point $z_0 \in \mathbb{C} \setminus \Gamma$.

A singular integral operator with a Carleman shift is defined to be the operator of the form

$$(M\varphi)(t) = \sum_{k=0}^{n-1} \left\{ a_k(t)\varphi\big(\alpha_k(t)\big) + \frac{b_k(t)}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - \alpha_k(t)}\, d\tau \right\} \tag{1.2}$$

where $a_k(t)$ and $b_k(t)$ are given functions on the contour Γ. For a limited Lyapunov contour Γ the Noether theory of operators of the form (1.2), build in [10,18], and the algebra, generated by operators of the form (1.2), are considered in papers [2,3] and others. In the mentioned works the complete continuity of operators $T_k = W^k S W^{-k} - S$, where $(W\varphi)(t) = \varphi(\alpha(t))$ and S is the operator of singular integration along Γ

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - \alpha_k(t)}\, d\tau \quad (t \in \Gamma),$$

was essentially used.

The situation is absolutely different if Γ is an unlimited contour. In this case [6,17] operator W is unbounded, generally speaking, in spaces L_p; instead of the operator W it is necessary to consider [12,15] the operator

$$(V\varphi)(t) = \left(\frac{\alpha(t) - z_0}{t - z_0}\right)^\lambda \varphi\big(\alpha(t)\big) \quad \left(\frac{1}{p} < \lambda < 1 + \frac{1}{p}\right) \tag{1.3}$$

and, as was shown in [14], for $\lambda \neq 1$ the operators $V^k S V^{-k} - S$ are not completely continuous. The difficulties of the research of operators (1.2) along an unlimited contour consist in these facts.

In the present paper on the base of results of works [12–16] the least subalgebra Σ of the algebra $L(L_p(\Gamma, \rho))$, containing all operators of the form (1.2) with piecewise continuous coefficients, is studied. It is necessary to consider separately the case, when α preserves the orientation on Γ, and the case, when α reverses the orientation. The algebra Σ contains the set Σ_0 of all sums of compositions of operators of the form (1.2), and also operators, which are limits (in the sense of convergence by the norm of operators) of a sequence of operators from Σ_0. The research of the set Σ_0 is based on the suggested by I. Gohberg and N.Y. Krupnik [4] method of the study of "complicated" operators, which allows to receive necessary and sufficient conditions of Noetherian property of operators from Σ. In the paper the existence of such isomorphism between Σ and some algebra $\mathfrak{A}$ of singular operators with a Cauchy kernel that an arbitrary operator from Σ and its image are simultaneously Noetherian or not Noetherian is proved. It allows to introduce the concept of a symbol for all operators from Σ and, using known results for algebra $\mathfrak{A}$ (see [14]), in terms of a symbol to receive conditions of Noetherian property for all operators from Σ, including $\Sigma \setminus \Sigma_0$. Through the symbol the index of operators $A \in \Sigma$ can be also expressed.

The set of values of the determinant of a symbol $A(t, \mu)$ represents a closed continuous curve, which can be oriented in a natural way. The index of this curve (i.e. the number of turns about the origin), taken with opposite sign, is equal to the index of the operator A.

We adopt the following plan:

2. Symbol of singular integral operators without a shift along an unlimited contour,
3. The condition of Noetherian property of operators from Σ. The case of a closed contour,
4. The condition of Noetherian property of operators from Σ. Case of an unclosed contour,
5. Example.

2. Symbol of singular integral operators without a shift along an unlimited contour

Let Γ be an unlimited contour. We shall name Γ admissible, if the contour $\overline{\Gamma} = \gamma(\Gamma)$, $\gamma(\Gamma) = (t - z_0)^{-1}$ $(z_0 \notin \Gamma)$ is a simple closed or unclosed Lyapunov contour. Let $\mathfrak{A}$ be the algebra, generated by singular integral operators of the form

$$A = cI + dS, \tag{2.1}$$

where $c(t)$ and $d(t)$ are piecewise continuous matrix-functions of order n and S is a matrix-operator of singular integration along Γ. Our used methods for researching operators from algebra Σ assume the use of the Noether theory of operators from $\mathfrak{A}$. In this section we shall establish some results and formulate them so that it would be possible to use them conveniently in the case of an unlimited contour.

Let $t_1, t_2, \dots, t_{n-1}$ be some various points on the unlimited curve Γ, $t_m = \infty$; ρ, β_0, $\beta_2, \dots, \beta_{m-1}, \beta_m = \sum_{k=1}^{m-1} \beta_k + p - 2$ be real numbers, satisfying the relations $-1 < \beta_k < p - 1$, $k = 1, 2, \dots, m$, and

$$\rho(t) = \prod_{k=1}^{m-1} |t - t_k|^{\beta_k}. \tag{2.2}$$

Let us introduce the following notations: $L_p(\Gamma, \rho)$ is the Banach space L_p on the contour Γ with weight $\rho(t)$; $L_p^n(\Gamma, \rho)$ is the Banach space of n-dimensional vector functions $f = \{f_i\}_{i=1}^n$ with components $f_i \in L_p(\Gamma, \rho)$, $\Lambda_n(\Gamma)$ is the set of all matrix-functions $F(t)$ of order n, continuous at each point of the contour Γ, except, possibly, a finite number of points, at which these functions are continuous from the left and have finite limits from the right. For further it is convenient to introduce the operators $P = \frac{I+S}{2}$ and $Q = I - P$. Then a usual singular operator $A = cI + dS$ in the space $L_p^n(\Gamma, \rho)$ can be represented as $A = aP + bQ$, where $\alpha = c + d$ and $b = c - d$ $(a, b \in \Lambda_n(\Gamma))$.

To the operator A its symbol will be assigned (see [4,8]). Let introduce some notations, necessary for the definition of a symbol and let us assume for the beginning that the contour is closed. By $\theta = \theta(t)$, $f(t, \mu)$ and $h(t, \mu)$, $t \in \overline{\Gamma}$, $(0 \leqslant \mu \leqslant 1)$ we designate the following functions:

$$\theta(t) = \begin{cases} \pi - \frac{2\pi(1-\beta_k)}{p}, & \text{if } t = t_k \ (k = 1, \dots, m), \\ \pi - \frac{2\pi}{p}, & \text{if } t \in \overline{\Gamma} \setminus \{t_1, \dots, t_m\}, \end{cases}$$

$$f(t, \mu) = \begin{cases} \frac{\sin\theta\mu \exp(i\theta\mu)}{\sin\theta \exp i\theta}, & \text{if } \theta \neq 0, \\ \mu, & \text{if } \theta = 0, \end{cases}$$

and $h(t,\mu)=\sqrt{f(t,\mu)(1-f(t,\mu))}$, respectively.

The symbol of an operator A is defined to be a matrix-function $A(t,\mu)$ ($t\in\overline{\Gamma}, 0\leqslant\mu\leqslant 1$) of order $2n$, defined by the equality

$$A(t,\mu)=\left\|\begin{matrix} f(t,\mu)a(t+0)+(1-f(t,\mu))a(t) & h(t,\mu)(b(t+0)-b(t)) \\ h(t,\mu)(a(t+0)-a(t)) & f(t,\mu)b(t)+(1-f(t,\mu))b(t+0)\end{matrix}\right\|. \tag{2.3}$$

If Γ is unclosed, let t_1 be its starting point and $t_m=\infty$ its end point. If $a\in\Lambda_n(\Gamma)$, then we set

$$a(t_1)=\lim_{\substack{t\to t_1\\ t\in\Gamma}} a(t_1)\quad\text{and}\quad a(\infty)=\lim_{\substack{t\to\infty\\ t\in\Gamma}} a(t).$$

As all points $t\in\Gamma\setminus\{t_1,\infty\}$ the symbol $A(t,\mu)$ of the operator $A=aP+bQ$ will be defined by the equality (2.3). At points t_1 and $t=\infty$ we set

$$A(t_1,\mu)=\left\|\begin{matrix} f(t_1,\mu)a(t_1)+(1-f(t_1,\mu)) & h(t_1,\mu)(b(t_1)-1) \\ h(t_1,\mu)(a(t_1)-1) & f(t_1,\mu)+(1-f(t_1,\mu))b(t_1)\end{matrix}\right\|, \tag{2.4}$$

$$A(\infty,\mu)=\left\|\begin{matrix} f(\infty,\mu)+(1-f(\infty,\mu))a(\infty) & h(\infty,\mu)(1-b(\infty)) \\ h(\infty,\mu)(1-a(\infty)) & f(\infty,\mu)b(\infty)+(1-f(\infty,\mu))\end{matrix}\right\|, \tag{2.5}$$

where, remind, in the definition of functions f and h with $t=t_m=\infty$ the number $\beta_m=\sum_{k=1}^{m-1}\beta_k$. The symbol $R(t,\mu)$ of any operator $R\in\mathfrak{A}$ is constructed from symbol of the operator $A=aP+bQ$. This symbol will be written in the form

$$A(t,\mu)=\left\|\begin{matrix} a_{11}(t,\mu) & a_{12}(t,\mu) \\ a_{21}(t,\mu) & a_{22}(t,\mu)\end{matrix}\right\|, \tag{2.6}$$

where $a_{ij}(t,\mu)$ are blocks of dimension n.

Theorem 2.1. *In order that the operator R ($\in\mathfrak{A}$) be Noetherian in the space $L_p^n(\Gamma,\rho)$ it is necessary and sufficient that the condition*:

$$\det R(t,\mu)\neq 0\quad (t\in\overline{\Gamma},\ 0\leqslant\mu\leqslant 1) \tag{2.7}$$

holds. If the condition (2.7) *is satisfied, then*

$$\operatorname{Ind} A=-\frac{1}{2\pi}\left\{\arg\frac{\det R(t,\mu)}{\det a_{22}(t,0)\det a_{22}(t,1)}\right\}_{\substack{t\in\Gamma\\ 0\leqslant\mu\leqslant 1}}. \tag{2.8}$$

The number $\frac{1}{2\pi}\{\arg g(t,\mu)\}_{(t,\mu)\in\Gamma\times[0,1]}$ in the right-hand side of the equality (2.8) corresponds to the number of turns counter-clockwise of the curve $g(t,\mu)$ around the point $z=0$ in the complex plane.

The symbol of singular integrated operators can get a more perfect form in the case, when coefficients of operators are Noetherian. Really, if Γ is closed, then the symbol of the operator $A=\alpha P+bQ$ is determined by the equality

$$A(t,\mu)=\left\|\begin{matrix} a(t) & 0 \\ 0 & b(t)\end{matrix}\right\|.$$

Let Γ be an unclosed unlimited arc. As weight we shall take the function $\rho(t) = |t - t_1|^\beta$. Let us designate by $\Lambda = \Lambda(\Gamma)$ the spatial closed unlimited curve ($\Lambda \subset \overline{R^3}$), consisting of all points (x, y, z), satisfying the relations

$$x + iy \in \overline{\Gamma}, \quad -1 \leqslant z \leqslant 1, \quad (1 - z^2)(x + iy - t_1) = 0.$$

In other words, the curve $\Lambda(\Gamma)$ consists of two copies of the curve Γ, located in the planes $z = 1$ and $z = -1$, and a straightline segment, parallel to the z-axis and passing through the beginning of the unclosed contour Γ.

The contour Γ is oriented so that in the plane $z = 1$ the direction along $\Lambda(\Gamma)$ coincides with the direction along Γ (i.e. from the point t_1), and in the plane $z = -1$ is opposite. For the space L_p $(\Gamma, |t - t_1|^\beta)$ $(-1 < \beta < p - 1)$ we shall designate by $\Omega_{p,\rho}$ the function defined on $\Lambda(\Gamma)$ by the following equalities (see [8]):

$$\Omega_{p,p}(t, z) = \begin{cases} \frac{z(1+d^2) - i\varepsilon(1-z^2)d}{1+z^2d^2}, & \text{for } t = t_1, \infty, \\ z, & \text{for } t \in \overline{\Gamma} \setminus \{t_1, \infty\}, \end{cases}$$

where $d = \cot \pi \frac{(1+\beta)}{p}$, $\varepsilon = 1$ for $t = t_1$ and $\varepsilon = -1$ for $t = \infty$. The symbol of the operator $A = \alpha I + bS$ will be defined to be the following matrix-function

$$A(t, z) = a(t) + \Omega_{p,p}(t, z)b(t). \tag{2.9}$$

Theorem 2.2. *Let a and b be continuous matrix-functions on $\overline{\Gamma}$ of order n. In order that the operator $A = \alpha P + bQ$ be Noetherian in the space $L_p^n(\Gamma, \rho)$ it is necessary and sufficient that the condition*:

$$\det\big(a(t) + \Omega_{p,p}(t, z)b(t)\big) \neq 0 \quad \big(t, z \in \Lambda(\Gamma)\big)$$

be satisfied. If this condition is verified, then

$$\operatorname{Ind} A = -\operatorname{Ind} \det\big(a(t) + \Omega_{p,p}(t, z)b(t)\big).$$

The last formulas are in some sense more obvious and easier to check.

Let us consider the following example. Let $\Gamma = [0, \infty)$ and $\rho(x) = x^\beta$ $(-1 < \beta < p - 1)$. Let us set $z = (e^{2\pi\xi} - 1)/(e^{2\pi\xi} + 1)$ $(\xi \in \mathbb{R})$ assuming it to be continuous on $\overline{\mathbb{R}}$ by the continuity. Then (see [9]) the symbol of an operator A can be represented as

$$A(x, \xi) = a(x) + b(x)\psi(x, \xi),$$

where

$$\psi(x, \xi) = \begin{cases} \frac{e^{2\pi(\xi + i\gamma)} + 1}{e^{2\pi(\xi + i\gamma)} - 1}, & \text{for } x = t_1, \infty, \\ \frac{e^{2\pi\xi} - 1}{e^{2\pi\xi} + 1}, & \text{for } 0 \leqslant x \leqslant \infty, \end{cases}$$

and $\gamma = \frac{1+\beta}{p}$. Note that the domain of the symbol $A(x, \xi)$ is the boundary X of the set $[0, \infty] \times [-\infty, \infty]$.

The following theorem plays an important role further on.

Theorem 2.3. *Let δ_k be some real numbers and $h(t) = \prod_{k=1}^{m-1} (t - t_k)^{\delta_k}$. If*

$$-\frac{1+\beta_k}{p} < \delta_k < 1 - \frac{1+\beta_k}{p},$$
$$-\frac{1+\sum_{k=1}^{m-1}\beta_k}{p} < \delta_m = -\sum_{k=1}^{m-1}\delta_k < 1 - \frac{1+\sum_{k=1}^{m-1}\beta_k}{p}, \tag{2.10}$$

then the operator $H = h(t)Sh^{-1}(t)I$ belongs to the algebra $\mathfrak{A}$ and its symbol $H(t,\mu)$ has the form

$$H(t,\mu) = \left\| \begin{matrix} E_n & U(t,\mu)E_n \\ 0 & -E_n \end{matrix} \right\| \tag{2.11}$$

where E_n is a unity matrix of dimension n and

$$U(t,\mu) = \begin{cases} \frac{4ih(t,\mu)\sin(\pi\delta_k\exp(\pi i\delta_k))}{2if(t_k,\mu)\sin(\pi\delta_k)\exp(\pi i\delta_k)+1}, & \text{for } t = t_k,\ k = 1,\dots,m, \\ 0, & \text{for } t \in \Gamma \setminus \{t_1,\dots,t_m\}. \end{cases}$$

3. The condition of Noetherian property of operators from Σ. The case of a closed contour

In this section we will construct a such homeomorphism Δ from algebra Σ to algebra $\mathfrak{A}$ such that the operators $A \in \Sigma$ and $\Delta(A) \in \mathfrak{A}$ are simultaneously Noetherian or not Noetherian with the Noether condition $\operatorname{Ind} A = \frac{1}{n}\operatorname{Ind}\Delta(A)$. The symbol of the operator A is defined as the symbol of the respective operator $\Delta(A) \in \mathfrak{A}$. We shall establish that an operator A is Noetherian if and only if the determinant of its symbol is not equal to zero. The index of the operator A can be also expressed through its symbol.

Let $\alpha(t)$ be the shift that satisfies the conditions (1.1) and $\alpha_n(t) \cong t$. We shall designate $t_n = \infty$ and we shall calculate the iteration $t_j = \alpha_j(t_n)$ $(j = 1,2,\dots,n-1)$. Let us introduce the space $L_p(\Gamma,\rho)$ with weight

$$\rho(t) = \prod_{j=1}^{n-1} |t-t_j|^{\frac{p\theta-2}{n}} \tag{3.1}$$

where λ is some real number, satisfying the condition

$$\frac{n-2}{p(n-1)} < \lambda < \frac{n(p+1)-2}{p(n+1)}. \tag{3.2}$$

It is obvious that $-1 < \frac{p\lambda-2}{n} < p-1$ and $-1 < \frac{(p\lambda-2)(n-1)}{n} < p-1$.

The suppositions made about the weight $\rho(t)$ ensure (see [7]) the boundedness of the operator S and

$$(V\varphi)(t) = \left(\frac{\alpha(t)-z_0}{t-z_0}\right)^{\lambda}\varphi\big(\alpha(t)\big)$$

in the space $L_p(\Gamma,\rho)$. Let us designate $(A_k\varphi)(t) = \tilde{a}_k(t)\varphi(t) + \tilde{b}_k(t)(S\varphi)(t)$, where

$$\tilde{a}_k(t) = \left(\frac{\alpha_{n-k}(t)-z_0}{t-z_0}\right)^{\lambda} a_k\big(\alpha_{n-k}(t)\big),$$
$$\tilde{b}_k(t) = \left(\frac{\alpha_{n-k}(t)-z_0}{t-z_0}\right)^{\lambda} b_k\big(\alpha_{n-k}(t)\big),$$

then the operator M, defined by the equality (1.2), takes the form

$$M = A_0 + VA_1 + \cdots + V^{n-1}A_{n-1}. \tag{3.3}$$

Everywhere we shall consider that the functions $\tilde{a}_k(t)$ and $\tilde{b}_k(t) \in \Lambda_1(\Gamma)$. The case when coefficients $\tilde{a}_k, \tilde{b}_k \in \Lambda_n(\Gamma)$ can be absolutely similarly investigated. The following theorem thus is valid [15].

Theorem 3.1. *The operator M is bounded in the space $L_p(\Gamma, \rho)$.*

We pass to the construction of the homomorphism Δ specified above. Let us designate by $\mathfrak{B}^n_{p,\rho}(\Gamma)$ the Banach algebra of all operators, which operate in the space $L^n_p(\Gamma, \rho)$, and by t^n the two-sided ideal of algebra $\mathfrak{B}^n_{p,\rho}(\Gamma)$, which consists of all completely continuous operators.

Let us take an arbitrary point $\tau_0$on the contour Γ and calculate the iterations $\tau_k = \alpha_k(\tau_0)$, $k = 1, 2, \ldots, n-1$. As was established in [10], we can assume points τ_k ordered in positive direction on Γ. The contour Γ is divided into n not intersecting arcs: $(\tau_0, \tau_1), (\tau_1, \tau_2), \ldots, (\tau_{n-2,}\tau_{n-1})$, $(\tau_{n-1,}\tau_{0,})$. Let us define the function $\omega(t)$ on (τ_0, τ_1) as follows: the function $\omega(t)$ is equal to zero outside the arc $[\tau_0, \tau_1]$ and $\omega(\tau_0) = 1$, $\omega(\tau_1) = \varepsilon$ $(\varepsilon = e^{\frac{2\pi i}{n}})$; on $[\tau_0, \tau_1]\omega(t)$ is continuous, bounded and different from zero. By $U(t)$ we shall designate the function

$$U(t) = \sum_{k=0}^{n-1} \varepsilon^k w\bigl(\alpha_{n-k}(t)\bigr) \quad (t \in \Gamma).$$

It is clear that the function $U(t)$ is continuous everywhere on Γ, bounded and different from zero, and satisfies the condition $U(t) - \varepsilon^{-1}U(\alpha(t)) = 0$ everywhere on Γ.

Let δ_{mj} be the Kronecker delta and consider the operators

$$R = \left\|\delta_{mj} U^{m-1} I\right\|^n_{m,j=1}; \qquad N = \left\|\varepsilon^{(m-1)(j-1)} V^{m-1}\right\|^n_{m,j=1};$$

which belong to the algebra $\mathfrak{B}^n_{p,\rho}(\Gamma)$ (here and below m is the row number, j is the column number). The operators R and N are invertible, and

$$R^{-1} = \left\|\delta_{mj} U^{1-m} I\right\|^n_{m,j=1}; \qquad N^{-1} = \frac{1}{n}\left\|\varepsilon^{(1-m)(j-1)} V^{1-m}\right\|^n_{m,j=1}.$$

Let us define the homeomorphism $\delta : \mathfrak{B}_{p,\rho}(\Gamma) \mapsto \mathfrak{B}^n_{p,\rho}(\Gamma)$ as follows. If $A \in \mathfrak{B}_{p,\rho}(\Gamma)$, then we assume

$$\delta(A) = NR\Theta(A)R^{-1}N^{-1}, \tag{3.4}$$

where $\Theta(A) = \|\delta_{mj} A\|^n_{m,j=1} \in \mathfrak{B}^n_{p,\rho}(\Gamma)$.

Lemma 3.1. *The operators A and $\delta(A)$ are Noetherian with Noether condition*

$$\operatorname{Ind} A = \frac{1}{n} \operatorname{Ind} \delta(A).$$

For a proof see [6].

Theorem 3.2. *The contraction Δ of homeomorphism δ to algebra Σ is also a homeomorphism*

$$\Delta : \Sigma \longmapsto \mathfrak{A}$$

and, if M is the operator (3.5)*, then*

$$\Delta(M) \simeq \widetilde{M}, \tag{3.5}$$

where $\widetilde{M} = \beta(t)I + \gamma(t)\widetilde{S}$, $\beta(t) = \|\tilde{a}_{-r+k+n}(\alpha_{k-1}(t))\|_{r,k=1}^{n}$, $\gamma(t) = \|\tilde{b}_{-r+k+n}(\alpha_{k-1}(t))\|_{r,k=1}^{n}$ *and* $\widetilde{S}$ *is defined in* $L_p^n(\Gamma, \rho)$ *by the equality*

$$\widetilde{S} = \left\| \begin{matrix} S & & & 0 \\ & (t-t_1)^{1-\lambda} S (t-t_1)^{\lambda-1} & & \\ & & \ddots & \\ 0 & & & (t-t_{n-1})^{1-\lambda} S (t-t_{n-1})^{\lambda-1} \end{matrix} \right\|.$$

N.B. Here and further $K_1 \simeq K_2$ means that the operator $K_1 - K_2$ is completely continuous.

Proof. It is sufficient to find out the way in which the homomorphism Δ acts on operators of the form (3.3). Let $M_j = \sum_{r=0}^{n-1} \varepsilon^{rj} V^r A_r$ $(j = 0, 1, \ldots, n-1)$, $A_r = \tilde{a}_r I + \tilde{b}_r S$, $L = \|\delta_{mj} M_{j-1}\|_{m,k=1}^{n}$ $(M_0 = M)$ and $\Phi = \|V^{k-1} A_{-m+k+n} V^{1-k}\|_{m,k=1}^{n}$ $(A_{n+i} = A_i)$.

Then, as is known, the following equality is satisfied:

$$L = N^{-1} \Phi N. \tag{3.6}$$

It is easy to check that $U^j M \simeq M_j U^j$. Therefore

$$L \simeq R \Theta(M) R^{-1}. \tag{3.7}$$

As (see [15]) $V^r S V^{-r} \simeq (t-t_r)^{1-\lambda} S (t-t_r)^{\lambda-1}$ $(r = 1, \ldots, m-1)$, then $\Phi \simeq \widetilde{M}$. From here it follows that

$$\Delta(M) \simeq \widetilde{M}.$$

The theorem is proved. □

The following theorem results from Lemma 3.1 and Theorem 3.2:

Theorem 3.3. *If A is an arbitrary operator from algebra Σ, then the operators A and $\Delta(A)$ are Noetherian or not Noetherian simultaneously, with Noether condition*

$$\operatorname{Ind} A = \frac{1}{n} \operatorname{Ind} \Delta(A).$$

As the operator $\Delta(A) \in \mathfrak{A}$ and for operators from algebra $\mathfrak{A}$ the concept of symbol was introduced, then a symbol of an operator $A \in \Sigma$ can be naturally defined to be the symbol of the operator $\widetilde{A} = \Delta(A)$. Thus, the symbol of an operator $A \in \Sigma$ is the matrix-function $A(t, \mu) = \widetilde{A}(t, \mu)$ of order $2n$. In particular, the symbol of the operator $M = A_0 + V A_1 + \cdots + V^{n-1} A_{n-1}$ $(A_j = \tilde{a} I + \tilde{b} S)$ according to the formulas (2.3) and (2.10) has the form

$$A(t,\mu) = \left\| \begin{matrix} f(t,\mu)\beta(t+0) + (1 - f(t,\mu)\beta(t)) & h(t,\mu)(\beta(t+0) - \beta(t)) \\ h(t,\mu)(\beta(t+0) - \beta(t)) & f(t,\mu)\beta(t) + (1 - f(t,\mu)\beta(t+0)) \end{matrix} \right\|$$
$$+ \left\| \begin{matrix} f(t,\mu)\gamma(t+0) + (1 - f(t,\mu)\gamma(t)) & h(t,\mu)(\gamma(t+0) - \gamma(t)) \\ h(t,\mu)(\gamma(t+0) - \gamma(t)) & f(t,\mu)\gamma(t) + (1 - f(t,\mu)\gamma(t+0)) \end{matrix} \right\|$$
$$\times \widetilde{S}(t,\mu),$$

where

$$\widetilde{S}(t,\mu)=\left\|\begin{matrix} 1 & & 0 & 0 & & 0 \\ \dots & \dots & \dots & & U_1(t,\mu) & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & & 1 & 0 & & U_{n-1}(t,\mu) \\ \dots & \dots & \dots & -1 & & 0 \\ & 0 & & \dots & \dots & \dots \\ & & & 0 & & -1 \end{matrix}\right\|$$

and because of the formula (2.11), the functions $U_j(t,\mu)$ are defined by the equalities

$$U_j(t,\mu)=\begin{cases} \frac{-4ih(t_j,\mu)\sin(\pi\lambda)\exp(-\pi i\lambda)}{-2if(t_j,\mu)\sin(\pi\lambda)\exp(-\pi i\lambda)+1}, & \text{if } t=t_j, \\ \frac{4ih(t_n,\mu)\sin(\pi\lambda)\exp(\pi i\delta_k)}{2if(t_n,\mu)\sin(\pi\lambda)\exp(\pi i\lambda)+1}, & \text{if } t=t_n, \\ 0, & \text{if } t\in\Gamma\setminus\{t_1,\dots,t_n\}. \end{cases}$$

If

$$M=\sum_{j=1}^{m} M_{j1}M_{j2}\dots M_{jr}, \tag{3.8}$$

where M_{ji} are operators of the form (3.3), then

$$M(t,\mu)=\sum_{j=1}^{m} M_{j1}(t,\mu)M_{j2}(t,\mu)\dots M_{jr}(t,\mu), \tag{3.9}$$

where $M_{ji}(t,\mu)$ is the symbol of operator M_{ji}. The set of all operators of the form (3.8) will be designated by Σ_0. Using the method described in [4] one can prove the following theorem:

Theorem 3.4. *If the operator M belongs to the algebra Σ_0 and $M(t,\mu)=\|a_{jk}(t,\mu)\|_{j,k=1}^{2}$ is its symbol, then*

$$\max_{t\in\Gamma;\ 0\leqslant\mu\leqslant 1}\left|a_{jk}(t,\mu)\right|\leqslant\inf_{T\in\mathfrak{F}}\|A+T\|, \tag{3.10}$$

where $\mathfrak{F}$ is the set of all completely continuous operators in the space $L_p(\Gamma,\rho)$.

Corollary 3.1. *The symbol of the operator $M\in\Sigma_0$ does not depend on representation of an operator in the form*

$$M=\sum_{j=1}^{m} M_{j1}M_{j2}\dots M_{jr}.$$

Corollary 3.2. *The mapping $M\mapsto M(t,\mu)$ of the algebra Σ_0 to the algebra of symbols is an algebraic homeomorphism, whose kernel contains the set of all completely continuous operators from algebra Σ_0.*

Let $A\in\Sigma\setminus\Sigma_0$ and $A=\lim_{n\to\infty}A_n$, where $A_n\in\Sigma_0$. In virtue of the relation (3.10), symbols of operators A_n converge uniformly to some continuous matrix-function $A(t,\mu)$, which does not depend on the choice of a sequence $\{A_n\}_1^\infty$, and which we shall name the symbol of the operator A.

From Theorem 3.3 and from a property of the symbol of algebra $\mathfrak{A}$ follows

Theorem 3.5. *In order that the operator $A \in \Sigma$ be Noetherian in the space $L_p(\Gamma, \rho)$ it is necessary and sufficient that its symbol*

$$A(t,\mu) = \begin{Vmatrix} a_{11}(t,\mu) & a_{12}(t,\mu) \\ a_{21}(t,\mu) & a_{22}(t,\mu) \end{Vmatrix}$$

be non-degenerate: $\det A(t,\mu) \neq 0$ $(t \in \Gamma,\ 0 \leqslant \mu \leqslant 1)$. *If this condition is fulfilled, there exists a two-sided regulator of the operator, belonging to the algebra Σ and*

$$\operatorname{Ind} A = -\frac{1}{2n\pi}\left\{\arg \frac{\det A(t,\mu)}{\det a_{22}(t,0)\det a_{21}(t,1)}\right\}_{t\in\Gamma,\ 0\leqslant\mu\leqslant 1}.$$

N.B. If A is Noetherian, then there exists a linear bounded operator B such that $I - AB$ and $I - BA$ are completely continuous; B is referred to as a two-sided regulator of the operator A.

4. The condition of Noetherian property of operators from Σ. The case of a non-closed contour

Continuous coefficients. Let us consider at first, the case, when the algebra Σ is generated by operators S, V, and operators of multiplication by continuous functions. Let Γ be an unclosed contour with the starting point $t = t_1$ and the extremity $t = \infty$ and $\alpha : \Gamma \mapsto \Gamma$ satisfies the condition $\alpha(\alpha(t)) \equiv t$, then it is obvious that $\alpha(t_1) = \infty$ and $\alpha(\infty) = t_1$. Let

$$\rho(t) = |t - t_1|^\beta \quad \left(\beta = \frac{p\lambda - 2}{2},\ \frac{1}{p} < \lambda < 1 + \frac{1}{p}\right).$$

Lemma 4.1. *Each operator $A \in \Sigma$ can be represented as a sum $A = A_1 + A_2V$ $(A_1, A_2 \in \mathfrak{A})$ and it is uniquely determined up to completely continuous summands.*

Proof. Obviously, it is enough to show that the set of operators of the form

$$A_1 + A_2V, \tag{4.1}$$

where A_1 and A_2 run algebra $\mathfrak{A}$, is a Banach algebra and from the equality $A_1 + A_2V = 0$, $(A_1, A_2 \in \mathfrak{A})$ it follows that A_1, A_2 are completely continuous operators. From the equality $VSV = -(t - t_0)^{1-\lambda}S(t - t_0)^{\lambda-1} + T$ $(\in \mathfrak{A})$ and $V\alpha V = \alpha(\alpha(t))I$ it follows that the sum and the product of operators of the form (4.1) have the same form. Let us prove that this set is closed.

Let $A_1^{(n)} + A_2^{(n)}V$ converge uniformly to A. Let U be an invertible operator, built in [12], satisfying the condition $U^{-1}\ (A_1^{(n)} + A_2^{(n)}V)U = A_1^{(n)} - A_2^{(n)}V + T_n$. Then $2\widetilde{A}_1^{(n)} \mapsto \widetilde{A} + \widetilde{U}^{-1}\widetilde{A}\widetilde{U} = 2\widetilde{A}_1 \in \widetilde{\mathfrak{A}} =\in \mathfrak{A}/\mathfrak{F}(L_p(\Gamma,\rho))$ is the factor-algebra in all completely continuous operators, acting in the space $L_p(\Gamma,\rho)$. Analogously, $\widetilde{A}_2^{(n)} \mapsto \widetilde{A}_2 \in \widetilde{\mathfrak{A}}$.

So, $\widetilde{A} = \widetilde{A}_1 + \widetilde{A}_2V$ and, hence, $A = A_1 + A_2V$ $(A_1, A_2 \in \mathfrak{A})$.Let $A_1 + A_2V = 0$, then $\widetilde{A}_1 + \widetilde{A}_2\widetilde{V} = 0$ and $\widetilde{A}_1 - \widetilde{A}_2\widetilde{V} = \widetilde{U}^{-1}\ (\widetilde{A}_1 + \widetilde{A}_2\widetilde{V})\widetilde{U}$, i.e., $A_1, A_2 \in \mathfrak{F}(L_p(\Gamma,\rho))$. □

Lemma 4.2. *Let $A, B, W \in L(\mathfrak{B})$ and $W^2 = I$, then the equality is satisfied* (*see* [10])

$$\begin{Vmatrix} I & W \\ I & -W \end{Vmatrix} \begin{Vmatrix} A & B \\ WBW & WAW \end{Vmatrix} \begin{Vmatrix} I & W \\ I & -W \end{Vmatrix} = 2\begin{Vmatrix} A + BW & 0 \\ 0 & A - BW \end{Vmatrix}. \tag{4.2}$$

The proof can be checked immediately (see [1]).

Theorem 4.1. *Let $A, B \in \mathfrak{A}$ and $(V\phi)(t) = (\frac{\alpha(t)-z_0}{t-z_0})^\lambda \phi(\alpha(t))$. The operator $R = A + BV$ is Noetherian in the space $L_p(\Gamma, \rho)$ if and only if the operator*

$$R_V = \begin{Vmatrix} A & B \\ VBV & VAV \end{Vmatrix} \tag{4.3}$$

is Noetherian in the space L_p^2. If $A + BV$ is Noetherian, then

$$\operatorname{Ind}(A + BV) = \frac{1}{2} \operatorname{Ind} \begin{Vmatrix} A & B \\ VBV & VAV \end{Vmatrix}. \tag{4.4}$$

Proof. Let us write the equality (4.2) for operators A, B, V. Observe that $U^{-1}(A - BV)U = A + BV + T$, hence, $A + BV$ is Noetherian only simultaneously with the operator $A - BV$ and $\operatorname{Ind}(A + BV) = \operatorname{Ind}(A - BV)$. Since extreme multiplicands in the left-hand side of (4.2) are invertible operators (their product is $2I$), all the statements of theorem result from the equality (4.2). Theorem is proved. □

Remark 4.1. The matrix-operator (4.3) is a singular operator with matrix-coefficients (without shift).

Corollary 4.1. *Let $A = aI + bS$, $B = cI + dS$, $a, b, c, d \in C(\overline{\Gamma})$, $R = A + BV$ and $(V\varphi)(t) = \frac{\alpha(t)-z_0}{t-z_0}\varphi(\alpha(t))$. Then,*

$$R_V = \begin{Vmatrix} A & B \\ VBV & VAV \end{Vmatrix} = \begin{Vmatrix} a & c \\ \tilde{c} & \tilde{a} \end{Vmatrix} I + \begin{Vmatrix} b & d \\ -\tilde{d} & -\tilde{b} \end{Vmatrix} S + T, \tag{4.5}$$

where $\tilde{f}(t) = f(\alpha(t))$ and T is a completely continuous operator in $L_p^2(\Gamma, \rho)$.

The symbol of the operator $R = A + BV$ will be defined to be the matrix $R_V(t, \mu)$, $t \in \overline{\Gamma}$, $0 \leqslant \mu \leqslant 1$, defined by the equalities (2.3), (2.4) and (2.5). From above reasonings and Theorems 2.1 and 4.1 results.

Theorem 4.2. *In order that the operator $R = A + BV$ be Noetherian in the space $L_p(\Gamma, \rho)$ it is necessary and sufficient that*

$$\det R_V(t, \mu) \neq 0 \quad (t \in \overline{\Gamma},\ 0 \leqslant \mu \leqslant 1).$$

If this condition is satisfied, then

$$\operatorname{Ind} R = \frac{1}{4\pi} \left\{ \arg \frac{\det R_V(t, \mu)}{\det a_{22}(t, 0) \det a_{22}(t, 1)} \right\}_{t \in \overline{\Gamma},\ 0 \leqslant \mu \leqslant 1}.$$

Remark 4.2. The symbol of the operator R can be defined by the equality (2.9) if substitute for the matrices a and b the respective matrices of the operator R_V. In this case Theorem 2.2 for the operator $R = A + BV$ is true.

Discontinuous coefficients. Theorems 4.1 and 4.2 generally speaking cannot be transferred to the case, when the coefficients $a, b, c, d \in \Lambda_1(\overline{\Gamma})$. To show this, it is enough to give an example of two singular operators A and B such that $A+BV$ is Noetherian, but $A-BV$ is not Noetherian. From Eq. (4.2) it results that the respective matrix operator is not Noetherian. The example for $\Gamma=[-1,1]$ or $\Gamma=\mathbb{R}$ and $\alpha(x)=-x$ are built in works [1,14]. After constructing the symbol of operators of the form $A+BV$ we can construct such examples for every Γ and $\alpha\in V(\Gamma)$ (see Section 5).

Let introduce some notations. We denote $\Sigma(\Gamma,\rho)$ the Banach subalgebra of the algebra $L(L_p(\Gamma,\rho))$ $(\rho(t)=|t-t_1^\beta|, \beta=\frac{p\lambda-2}{2})$, generated by an operator S, completely continuous operators and all operators of multiplication by function $a\in\Lambda_1(\Gamma)$. By $\Sigma(\Gamma,\rho,B)$ $(B\in L(L_p(\Gamma,\rho)))$ we shall designate the Banach subalgebra generated by all operators from $\Sigma(\Gamma,\rho)$ and by the operator B. Let Σ be any subalgebra of algebra $L(\mathfrak{B})$. By Σ_n we shall designate the subalgebra of algebra $L(\mathfrak{B}_n^n)$ consisting of all operators of the form $|A_{jk}|_{j,k=1}^n$, where $A_{jk}\in\Sigma$.

Definition 4.1. Two algebra $\mathfrak{A}_1$ $(\subset L(\mathfrak{B}_1))$ and $\mathfrak{A}_2$ $(\subset L(\mathfrak{B}_2))$ will be called equivalent [1] if there is an invertible operator $M\in L(\mathfrak{B}_1,\mathfrak{B}_2)$ such that the set of operators of the form MAM^{-1} $(A\in\mathfrak{A}_1)$ coincides with the algebra $\mathfrak{A}_2$.

Theorem 4.3. *Let Γ be some closed or unclosed contour in the complex plane and $\rho(t)=\prod_{k=1}^n|t-t_k|^{\beta_k}$ $(-1<\beta_k<p-1)$, then the algebra $\Sigma(\Gamma,\rho)$ is equivalent with the algebra $\Sigma(\Gamma)=\Sigma(\Gamma,1)$.*

Proof. Let $(M\phi)(t)=\rho^{1/p}(t)\phi(t)$. As $\|M\phi\|_{L_p(\Gamma)}=\|\phi\|_{L_p(\Gamma,\rho)}$, M maps isometrically the space $L_p(\Gamma,\rho)$ on $L_p(\Gamma)$. As $MaM^{-1}=aI$ for any function $a\in\Lambda_1(\Gamma)$, therefore it is enough to show that $MSM^{-1}\in\Sigma(\Gamma)$. The operator MSM^{-1} can be represented as

$$MSM^{-1}=f(t)\prod_{k=1}^n|t-t_k|^{\beta_k/p}S\prod_{k=1}^n|t-t_k|^{-\beta_k/p}f^{-1}(t)I$$

where $f(t)$ is different from zero and continuous everywhere on Γ with exception, perhaps, the points t_k. The numbers $\alpha_k=\beta_k/p$ satisfy the condition

$$-\frac{1}{p}<\alpha_k<1-\frac{1}{p},$$

therefore (see [12]) the operator $\prod_{k=1}^n|t-t_k|^{\beta_k/p}S\prod_{k=1}^n|t-t_k|^{-\beta_k/p}I$ belongs to the algebra $\Sigma(\Gamma)$. Theorem is proved. □

Remark 4.3. If Γ is unlimited, it is necessary to require that numbers β_k together with the condition $-1<\beta_k<p-1$ $(k=1,\dots,n)$ satisfy the condition $-1<\sum_{k=1}^n\beta_k<p-1$.

Corollary 4.2. *Any two algebras $\Sigma(\Gamma,\rho_1)$ and $\Sigma(\Gamma,\rho_2)$ are equivalent.*

Further on we shall consider the algebra $\Sigma(\Gamma,\rho;V)$, where $\rho(t)=|t-t_1|^\beta$ $(\beta=\frac{p\lambda-2}{2}, \frac{1}{p}<\lambda<1+\frac{1}{p})$ and

$$(V\phi)(t)=\left(\frac{\alpha(t)-z_0}{t-z_0}\right)^\lambda\quad\bigl(\alpha\in V(\Gamma)\bigr).$$

Remark that we can assume that $t_1 = 0$, as otherwise it can be achieved with the help of the invertible operator $H : L_p(\Gamma, \rho) \mapsto L_p(\widetilde{\Gamma}, \tilde{\rho})$, defined by the equality $(H\phi)(z) = \phi(z + t_1) \times ((H^{-1}\phi)(t) = \phi(t - t_1))$, where $\tilde{\rho}(t) = |z|^\beta$ and $\widetilde{\Gamma} = \{z = t - t_1\colon\ t \in \Gamma\}$. For the same reason, it is possible to consider that $z_0 \notin \Gamma$. Besides it is enough to study the algebra $\Sigma(\mathbb{R}^+, x^\beta; W)$, where

$$(W\varphi)(x) = \left(\frac{\alpha(x) + 1}{x + 1}\right)^\lambda \varphi\big(\alpha(x)\big) \quad \big(\alpha \in V(\mathbb{R}^+)\big).$$

and $\alpha \in V(\mathbb{R})$. Really, let $\gamma : \mathbb{R}^+ \mapsto \Gamma$ be a homeomorphic mapping from $\mathbb{R}^+$ on Γ with the following properties: $\gamma(0) = 1$, $\gamma(\infty) = \infty$ and $0 \neq \gamma'(x) \in H(\mathbb{R}^+)$. Obviously (see [5]), such a function exists. The operator M, defined by the equality

$$(M\varphi)(x) = \varphi\big(\gamma(x)\big),$$

is a linear bounded invertible operator, acting from $L_p(\Gamma, |t|^\beta)$ into $L_p(\mathbb{R}^+, x^\beta)$. Let us designate by ω the function $\omega(t) = \gamma^{-1}(t)$. Then $(M^{-1}\phi)(t) = \phi(\omega(t))$, $MSM^{-1} = S_+ + T$, where

$$(S_+\varphi)(x) = \frac{1}{\pi i}\int_0^\infty \frac{\varphi(y)}{y - x}\,dy \quad \text{and} \quad (T\varphi)(x) = \frac{1}{\pi i}\int_0^\infty \left(\frac{\gamma'(y)}{\gamma(y) - \gamma(x)} - \frac{1}{y - x}\right)\varphi(y)\,dy,$$

is completely continuous in $L_p(\mathbb{R}^+, x^\beta)$. It is easy to see that $M\alpha M^{-1} = \alpha(\gamma(x))I$ and $MVM^{-1}\varphi = (\frac{\alpha(\gamma(x))+1}{\gamma(x)+1})^\lambda \varphi(\omega \circ \alpha \circ \gamma)(x)$. Let us denote $\nu(x) = (\omega \circ \alpha \circ \gamma)(x)$. From properties of the functions γ, ω and α easily follows that $\nu \in V(\mathbb{R}^+)$. The operator MVM^{-1} can be represented as $MVM^{-1} = fW$, where $f(x) = (\frac{\alpha(\gamma(x))+1}{\gamma(x)+1})^\lambda (\frac{x+1}{\gamma(x)+1})^\lambda \in C(\overline{\mathbb{R}^+})$, is not equal to zero and $(W\varphi)(x) = (\frac{\nu(x)+1}{x+1})^\lambda \varphi(\nu(x))$. Thus $M\Sigma(\Gamma, |t|^\beta; V)M^{-1} = \Sigma(\mathbb{R}^+, x^\beta; W)$. So, further on we shall consider the algebra $\Sigma(\mathbb{R}^+, x^\beta; W)$ and we shall show that it is equivalent to some algebra generated by singular integral operators without a shift.

Theorem 4.4. *Let $\alpha \in V(\mathbb{R}^+)$,*

$$(W\varphi)(x) = \left(\frac{\alpha(x) + 1}{x + 1}\right)^\lambda \varphi\big(\alpha(x)\big), \tag{4.6}$$

$\rho(x) = x^\beta$ ($\beta = \frac{p\lambda - 2}{2}$, $\frac{1}{p} < \lambda < 1 + \frac{1}{p}$). The algebra $\Sigma(\mathbb{R}^+, \rho; W)$ is equivalent to the algebra $\Sigma(\Gamma_0, \rho_0; V_0)$, where $\Gamma_0 = [-1, 1]$, $\rho_0(t) = (1 - t^2)^\beta$ and $(V_0\varphi)(t) = \varphi(-t)$.

To prove this theorem, we need the following lemma.

Lemma 4.3. *Let $\alpha \in V(\mathbb{R}^+)$ and $\alpha(x_0) = x_0$, then there exists a homeomorphic map $\mu : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with the following properties*:

$$\mu(0) = 0, \quad \mu(x_0) = 1, \quad 0 \neq \mu'(x) \in H(\mathbb{R}^+), \tag{4.7}$$

$$(\mu^{-1} \circ \alpha \circ \mu)(x) = \frac{1}{x}. \tag{4.8}$$

Proof. Let $\tilde{\mu}(x) = \frac{x}{x_0}$ $(\tilde{\mu}^{-1}(x) = x \cdot x_0)$, then, obviously, the mapping $\nu = \tilde{\mu} \circ \alpha \circ \tilde{\mu}^{-1}$ belongs to the set $V(\mathbb{R}^+)$, and $\nu(1) = 1$. The mapping

$$\omega(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ \frac{1}{\nu(x)}, & \text{if } x \in [1,\infty], \end{cases} \tag{4.9}$$

satisfies the following conditions. There exists

$$\omega^{-1}(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ \gamma\left(\frac{1}{x}\right), & \text{if } x \in [1,\infty], \end{cases} \tag{4.10}$$

where $\omega'(x) \in H([0,1])$, $\omega'(x) \in H([1,\infty])$, $\omega'(x) \neq 0$ and , $\omega(1) = 1$. Let us show that $\omega'(x)$ is continuous at the point $x = 1$. This implies that $\omega'(x) \in H(\mathbb{R}^+)$. So, we have

$$\omega'(1-0) = 1, \qquad \omega'(1+0) = \lim_{x\to 1} -\frac{\nu'(x)}{\nu^2(x)} = -\frac{\nu'(1)}{\nu^2(1)} = 1,$$

since ν reverses the orientation on $\mathbb{R}^+$ and $\nu'(1) = -1$. The function $\mu = \tilde{\mu}^{-1} \circ \omega^{-1}$ satisfies the conditions (4.7). Let show that $(\mu^{-1} \circ \alpha \circ \mu)(x) = \frac{1}{x}$. We have $\mu^{-1} \circ \alpha \circ \mu = \omega \circ \tilde{\mu} \circ \alpha \circ \tilde{\mu}^{-1} \circ \omega^{-1} = \omega \circ \nu \circ \omega^{-1}$. Let $x \in [0,1]$, then $(\omega \circ \nu \circ \omega^{-1})(x) = \omega(\nu((\omega^{-1}(x)))) = \omega(\nu(\nu(\frac{1}{x}))) = \omega(\frac{1}{x}) = \frac{1}{x}$. Lemma is proved. □

Corollary 4.3. *The algebra $\Sigma(\mathbb{R}^+, x^\beta; W)$ is equivalent to the algebra $\Sigma(\mathbb{R}^+, x^\beta; W_0)$, where*

$$(W_0\varphi)(x) = \frac{1}{x^\lambda}\varphi\left(\frac{1}{x}\right). \tag{4.11}$$

Really, let us designate by M_1 the operator $(M\varphi)(x) = \varphi(\mu(x))$, where μ is the function from Lemma 4.3. Then the operator $M_1 S_+ M_1^{-1} - S_+$ is an integral operator with the kernel $\frac{1}{\pi i}(\frac{\mu'(y)}{\mu(y)-\mu(x)} - \frac{1}{y-x})$, and according to property (4.7), it is completely continuous in $L_p(\mathbb{R}^+, x^\beta)$. Besides, $M_1 \alpha M_1^{-1} = \alpha(\mu(x))I \in \Sigma(\mathbb{R}^+, x^\beta; W_0)$ for any function $\alpha \in \Lambda_1(\mathbb{R}^+)$. Let us show that $M_1 W M_1^{-1} \in \Sigma(\mathbb{R}^+, x^\beta; W_0)$. So, we have

$$\begin{aligned}
\left(M_1 W M_1^{-1}\varphi\right)(x) &= M_1 W \varphi\left(\mu^{-1}(x)\right) \\
&= M_1\left(\frac{\alpha(x)+1}{x+1}\right)^\lambda \varphi\left(\mu^{-1}\left(\alpha(x)\right)\right) \\
&= \left(\frac{\alpha(\mu(x))+1}{\mu(x)+1}\right)^\lambda \varphi\left(\frac{1}{x}\right) \\
&= f(x)(W_0\varphi)(x),
\end{aligned}$$

where the function $f(x) = [\frac{(\alpha(\mu(x))+1)x}{\mu(x)+1}]^\lambda$ is continuous and different from zero on $\overline{\mathbb{R}}^+$. This fact results from properties of functions μ and α (see [12]). Besides thereby, $M_1 W M_1^{-1} \in \Sigma(\mathbb{R}^+, x^\beta; W_0)$.

Proof of Theorem 4.4. Considering the above said, it remains to show that the algebra $\Sigma(\mathbb{R}^+, x^\beta; W_0)$ is equivalent with the algebra $\Sigma(\Gamma_0, \rho_0; V_0)$. Let M_2 be the operator acting by the rule:

$$(M_2\varphi)(t) = \frac{1}{1-t}\varphi\left(\frac{1+t}{1-t}\right). \tag{4.12}$$

The operator M_2 is linear, bounded and acting from $L_p(\mathbb{R}^+, x^\beta)$ into $L_p(\Gamma_0, h)$, where $\Gamma_0 = [-1, 1]$ and $h(t) = |1+t|^\beta |1+t|^{p-\beta-2}$. The operator M_2 is invertible, moreover:

$$\left(M_2^{-1}\Psi\right)(x) = \frac{2}{x+1}\Psi\left(\frac{x-1}{x+1}\right). \tag{4.13}$$

By simple calculations (we omit details) we make sure that $M_2 S_+ M_2^{-1} = S_0$, $M_2 \alpha M_2^{-1} = \tilde{\alpha} I$ ($\alpha \in \Lambda_1(\mathbb{R}^+)$) and $(M_2 W_0 M_2^{-1}\varphi)(t) = (\frac{1-t}{1+t})^{\lambda-1}\varphi(-t)$ where $\tilde{\alpha}(t) = \alpha(\frac{1-t}{1+t}) \in \Lambda_1(\Gamma_0)$ and $(S_0\varphi)(t) = \frac{1}{\pi i}\int_{-1}^{1}\frac{\varphi(\tau)}{\tau - t}\,d\tau$ ($t \in \Gamma_0$). Finally we will consider the operator

$$(M_3\varphi)(t) = (1-t)^{1-\lambda}\varphi(t),$$

which isometrically maps the space $L_p(\Gamma_0, h)$ into the space $L_p(\Gamma_0, \rho_0)$ ($\rho_0(t) = (1-t^2)^\beta$). It is easy to see that $M_3\tilde{\alpha}M_3^{-1} = \tilde{\alpha}I$, $M_2 S_0 M_2^{-1} = (1-t)^{1-\lambda}S_0(1-t)^{\lambda-1}I$ and $M_3 M_2 W_0 M_2^{-1} M_3^{-1} = V_0$, where $(V_0\varphi)(t) = \varphi(-t)$. Considering the condition $-\frac{1}{p} < \lambda < 1 - \frac{1}{p}$ and $\beta = \frac{p\lambda - 2}{2}$, from Theorem 2.2 we conclude that the operator $(1-t)^{1-\lambda}S_0(1-t)^{\lambda-1}I$ belongs to the algebra $\Sigma(\Gamma_0, \rho_0)$. In particular its symbol can be calculated with the help of (2.11). So, the algebra $\Sigma(\mathbb{R}^+, x^\beta; W_0)$ is equivalent to the algebra $\Sigma(\Gamma_0, \rho_0; V_0)$. Theorem is proved. □

Corollary 4.4. *The algebra $\Sigma(\Gamma, \rho; V)$ is equivalent with algebra $\Sigma_2(\widetilde{\Gamma}_0, \tilde{\rho}_0)$, where $\tilde{\rho}(t) = (1-t)^{\beta_t} t^{-1/2}$, $\widetilde{\Gamma}_0 = [0, 1]$.*

Really, from the proved above it follows that the algebra $\Sigma(\Gamma, \rho; V)$ is equivalent with the algebra $\Sigma(\Gamma_0, \rho_0; V_0)$. It remains to use results of works [1], in which it is proved that $\Sigma(\Gamma_0, \rho_0; V_0) \sim \Sigma(\widetilde{\Gamma}_0, \tilde{\rho}_0)$.

So, from Theorem 4.4 and results of work [1], we can conclude that there exists an invertible operator $X \in L(L_p(\Gamma, \rho), L_p^2(\widetilde{\Gamma}_0, \tilde{\rho}_0))$ such that $XAX^{-1} \in \Sigma_2(\widetilde{\Gamma}_0, \tilde{\rho}_0)$ for any operator $A \in \Sigma(\Gamma, \rho; V)$. Let us denote by $A(t, \mu)$ ($0 \leqslant t, \mu \leqslant 1$) the symbol of the operator $XAX^{-1} \in \Sigma_2(\widetilde{\Gamma}_0, \tilde{\rho}_0)$. The matrix-function $A(t, \mu)$ ($0 \leqslant t, \mu \leqslant 1$) of fourth order can be naturally called the symbol of the operator A. The symbol $A(t, \mu)$ of the operator $A \in \Sigma(\Gamma, \rho; V)$ we shall designate by $|a_{jk}(t, \mu)|_{j,k=1}^2$, where $a_{jk}(t, \mu)$ are matrix-functions of the second order. From properties of the symbols of operators from the algebra $\Sigma_2(\widetilde{\Gamma}_0, \tilde{\rho}_0)$ (see [4]) and arguments given above, results the following theorem:

Theorem 4.5. *In order that the operator $A \in \Sigma(\Gamma, \rho; V)$ be Noetherian in the space $L_p(\Gamma, \rho)$ it is necessary and sufficient that the determinant of its symbol can be separated from zero. If this condition is satisfied, then*

$$\operatorname{Ind} A - \frac{1}{2\pi}\left\{\frac{\arg\det A(t, \mu)}{\det a_{22}(t, 0)\det a_{22}(t, 0)}\right\}_{\substack{0 \leqslant t \leqslant 1 \\ 0 \leqslant \mu \leqslant 1}}.$$

5. Example

In this section an example of a singular operator with shift (reversing the orientation of the contour) is given, which shows that if the operator-coefficients have discontinuity points. Theorems 4.1 and 4.2, generally speaking, do not occur. For the sake of simplicity we will consider that $\Gamma = \mathbb{R}^+$, $\lambda = 1$ and $(W\varphi)(x) = \frac{1}{x}\varphi(\frac{1}{x})$. Let us consider the operator $R = I + \alpha\chi(x)S_+W$ in the space $L_z(\mathbb{R}^+)$, where $\alpha = \text{const}$, $\chi(x)$ $(x \in \mathbb{R}^+)$ is the characteristic function of the interval $[1, \infty]$ and S_+ is the operator of singular integration along $\mathbb{R}^+$. The operator R belongs to the algebra $\Sigma(\mathbb{R}^+; W)$. Its symbol looks like

$$R(t,\mu) = \begin{Vmatrix} a_{11}(t,\mu) & 0 \\ 0 & a_{22}(t,\mu) \end{Vmatrix},$$

where

$$a_{22}(t,\mu) = \begin{Vmatrix} 1 & -\alpha \\ 0 & 1 \end{Vmatrix}, \quad a_{11}(t,\mu) = \begin{Vmatrix} 1 & \alpha \\ 0 & 1 \end{Vmatrix} \quad \text{if } 0 < t < 1$$

and

$$a_{22}(t,\mu) = \begin{cases} \begin{Vmatrix} 1 & -\alpha(2\mu - 1) \\ 0 & 1 \end{Vmatrix} & \text{if } t = 1, \\ \begin{Vmatrix} 1 + i\alpha \sin \pi\mu & -\alpha \cos \pi\mu \\ 0 & 1 \end{Vmatrix} & \text{if } t = 0, \end{cases}$$

in the space $L_2(\mathbb{R}^+)$.

Consequently, $\det R(t,\mu) = 1$ if $t \neq 0$ and $\det R(t,\mu) = 1 + i\alpha \sin \pi\mu$. If consider $\alpha = -i$, then $\det R(t,\mu) \neq 0$ $(0 \leqslant t,\ \mu \leqslant 1)$; so, the operator $I - i\chi(x)S_+W$ is Noetherian in $L_2(\mathbb{R}^+)$. If consider $\alpha = i$, then $\det R(0, \frac{1}{2}) = 0$ and, therefore, the operator $I + i\chi(x)S_+W$ is not Noetherian in $L_2(\mathbb{R}^+)$.

Let $R_- = I - i\chi(x)S_+W$ and $R_+ = I + i\chi(x)S_+W$; as R_- is Noetherian, but R_+ is not Noetherian, then according to the equality (4.2), the operator R_W is not Noetherian in $L_2^2(\mathbb{R}^+)$. So, for the operator R Theorem 4.1 is not true. Example constructing is finished.

The analogous example for the space $L_2((-1,1))$ and $(W\varphi)(t) = \varphi(-t)$ was constructed in [1]. Note also that with the help of the operator M_2, determined by the equality (4.12), the constructed example can be reduced to the corresponding example from [1]. The case, when unlimited contour has corners [11], will be studied in other authors publications.

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